

The Linear Matching Method for Limit Loads and Shakedown Analysis of Portal Frames

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Abstract

We investigate the performance of a sequential programming method, based on the Linear Matching Method, for the direct evaluation of limit loads and shakedown limits for elastic-perfectly plastic portal frames subjected to complex histories of loading. This end is achieved by solving a sequence of linear problems defined with spatially varying moduli which relates properties of the yield condition to those of the linear problems. The method has been implemented in the commercial finite element package ABAQUS using the user defined procedures. It is shown that this method provides a sequence of upper bounds that monotonically reduces and converges to the least upper bound associated with the chosen class of displacement fields associated with FE mesh. A sequence of examples for a Von Mises yield condition demonstrates the ability of the procedure to produce a range of performance indicators for portal frames subjected to both cyclic vertical and horizontal loads.

Keywords

Shakedown Theory; Linear Matching Method; Cyclic Load; Perfect Plasticity; Portal Frame

Introduction

For the design of portal frames under monotonic/proportional loads, plastic methods of analysis are mainly used, which allows the engineer to analyse frames easily and design them economically. The basis of the plastic analysis method is the need to determine the load that can be applied to the frame so that the failure of the frame occurs as a mechanism by the formation of a number of plastic hinges within the frame. The collapse of the frame structures under these conditions is governed by three mechanisms known as beam collapse, sway collapse and a combined mechanism. However, for portal frame structures of an elastic-perfectly plastic material

subjected to variable and repeated loads, there is need for precise methods to predict the cyclic behaviour of these structures. Shakedown analysis provides significant advantages over other forms of analysis when a global understanding of deformation behaviour is required. These structures may fail either by fatigue, or by incremental collapse due to excessive deformations. On the contrary if the permanent deformations level off, these structures may come to steady state known as “shakedown”. This behaviour occurs when, after several excursions into the inelastic range, a residual stress field develops and all subsequent load cycles are resisted by elastic behaviour. The evaluation of the shakedown limits for such structures rely upon two component parts. The continuum is expressed in terms of either equilibrium stress fields or kinematically admissible strain rate fields within a discrete system. An optimal upper or lower bound to the shakedown load is then found by the application of a linear or non-linear programming method with the objective function derived from the upper and lower bound shakedown theorems. This has been the subject of extensive research work and a variety of techniques has been developed in the past two decades among others, for the behaviour of metal matrix composites subjected to cyclic loading, Ponter and Leckie, (1998), the performance of poro-elastic plastic structures with applications to dam design Cocchetti and Maier, (2000), the deformation of rolled surfaces Boulbibane and Collins, (2000), the inelastic behaviour of structures under variable repeated loading Weichert and Maier, (2001), the behaviour of portal frames made from a material that exhibits softening Cocchetti and Maier, (2003) and more recently by Barrera et al. (2011).

This paper describes the concept of the Linear

Matching Method and briefly summarised the basic formulae used for the evaluation of the upper bound shakedown load multipliers for portal frames subjected to variable and repeated loads. This concept was used by Ponter and Engelhardt (2000), Boulbibane and Ponter (2005) to derive a convergent programming method to evaluate the limit load of a body with general yield conditions that depend on the Von Mises effective stress and the hydrostatic pressure. Provided the convexity condition is satisfied, it was possible to define a sequence of linear problems where the upper bound functional monotonically reduces. The sequence then converges to the solution which corresponds to the absolute minimum of the functional, subject to constraints imposed by the class of strain rate histories under consideration. This method is unmistakably a kinematic approach. If one can be satisfied that all possible collapse mechanisms have been examined for incremental collapse, the lowest bound of the finite set is the actual load at the shakedown limit. Although the analyses presented here relate to a particular class of frame structures the approach can potentially be applied to a wide range of structural problems and types of material behaviour.

Portal Frame Analysis

Before we proceed to the presentation of the numerical results obtained by the Linear Matching Method, it would be pertinent here to explore the deformations that occur in a simple perfectly plastic portal frame under a vertical and horizontal monotonical loading. Figure 1 shows a simple frame problem with $h/l = 1.0$. The two loads H and V are applied together at their maximum values. There are five critical sections, as marked, where bending moments must be considered. The fully plastic moment is M_p for all members. A complete elastic-plastic analysis is not essential to the determination of the failure load, provided the correct mechanism can be found. The collapse of the frame under these conditions is governed by three mechanisms as shown in Fig. 1 (see for example Horne (1979)).

Using the standard methods of plastic structural analysis and under proportional loading one can demonstrate that for the beam mechanism which occurs when the vertical load is the dominant loading (Fig. 1b):

$$Vl / M_p = 8 \quad (1)$$

For the sway collapse, which occurs when the horizontal load is the dominant loading (Fig. 1c):

$$Hh / M_p = 4 \quad (2)$$

and for the combined mechanism (Fig. 1d), the limiting load is given by:

$$Vl / 2M_p + Hh / M_p = 6 \quad (3)$$

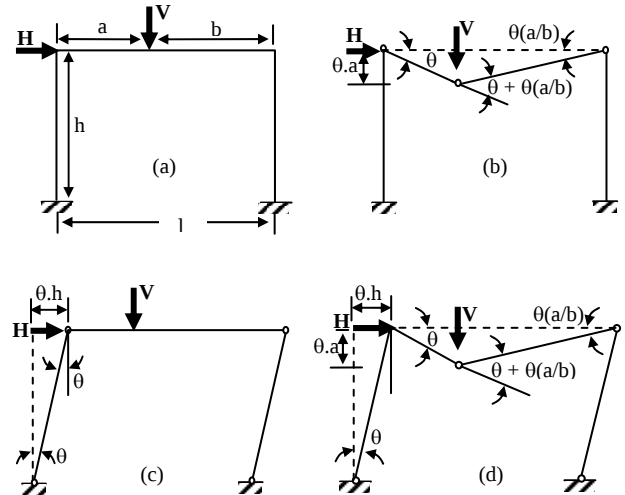


FIG. 1 COLLAPSE OF PORTAL FRAME UNDER COMBINATIONS OF VERTICAL AND HORIZONTAL LOADING

The combinations of V and H causing collapse according to the various mechanisms are shown graphically in Fig. 3. When more than one condition of loading can be applied to structure, it may not always be obvious which is critical. It is necessary then to perform separate calculations, one for each loading condition. However, when cyclic loads are applied to a structure which may vary in any way, we must explore all likely mechanisms to ensure that the correct mechanism has been chosen. The loading program may be a cycle that brings the moments at the critical sections one by one to the values required in one of the mechanisms of Fig. 1. It is hence necessary for variable repeated loading to derive a residual stress field that satisfy the inequality (4). The limiting values of these loads can be specified by a single parameter which represents the collapse load. The actual value of the collapse load is the lowest of all the upper bounds associated with the possible modes of collapse.

The Linear Matching Method

In this section we show that the shakedown limit load for a von Mises yield condition may be simulated by an incompressible linear solution with a spatial variation of the shear modulus. The method attempts to construct, as the limit of an iterative procedure, incompressible linear strain rate solution for the loads V and H . The iterative process results then in a

monotonically reducing upper bound:

$$\lambda_s \leq \lambda_{UB}^c = \frac{\int_0^{\Delta t} \int_V \sigma_{ij}^c(t) \dot{\varepsilon}_{ij}^c(t) dV dt}{\int_0^{\Delta t} \int_V \hat{\sigma}_{ij}(t) \dot{\varepsilon}_{ij}^c(t) dV dt} \quad (4)$$

that converges to the exact solution, if the elastic solutions $\hat{\sigma}_{ij}$ and linear solutions generated by the linear matching method are evaluated exactly, see Ponter and Engelhardt, (2000) and Ponter and Boulbibane, (2002) for more details. In the cases where the elastic solutions chosen are analytical, the upper bound converges to the least upper bound associated with the class of displacement fields described by the finite element mesh.

Consider the following problem. A body is composed of an isotropic elastic-perfectly plastic solid that satisfies the von Mises yield condition:

$$f(\sigma_{ij}) = \bar{\sigma} - \sigma_y \leq 0 \quad (5)$$

where

$$\bar{\sigma} = \sqrt{3/2(\sigma'_{ij}\sigma'_{ij})} \quad (6)$$

denotes the von Mises effective stress, $\sigma'_{ij} = \sigma_{ij} - \delta_{ij}\sigma_{kk}$ the deviatoric stress and σ_y is a uni-axial yield stress. The plastic strain rate, $\dot{\varepsilon}_{ij}^c$, is given by the associated flow rule in the form of the Prandtl-Reuss relationship,

$$\dot{\varepsilon}_{ij}^c = \frac{3}{2} \left(\frac{\bar{\varepsilon}}{\sigma_y} \right) \sigma'_{ij} \quad \text{and} \quad \dot{\varepsilon}_{kk}^c = 0 \quad (7)$$

where

$$\bar{\varepsilon} = \sqrt{2/3(\dot{\varepsilon}_{ij}^c \dot{\varepsilon}_{ij}^c)} \quad (8)$$

denotes the von Mises effective strain rate.

The upper bound on the shakedown limit (4) now simplifies to:

$$\int_0^{\Delta t} \int_V \sigma_y \bar{\varepsilon} (\dot{\varepsilon}_{ij}^c) dV dt = \lambda_{UB}^c \int_0^{\Delta t} \int_V (\hat{\sigma}_{ij} \dot{\varepsilon}_{ij}^c) dV dt \quad (9)$$

where $\lambda_{UB}^c \geq \lambda_s$, with λ_s the exact shakedown limit. A further simplification of (9) can be obtained when the history of loading produces a history of elastic stress which describes a polygon in stress space. In this case plastic strains only occur at m instances during the load cycle and we may replace the continuous strain rate history $\dot{\varepsilon}_{ij}^c$ by m discrete increments of plastic strain $(\Delta \varepsilon_{ij}^r, r=1, \dots, m)$ and the upper bound can be rewritten as:

$$\int_V \sum_{r=1}^m \sigma_y \bar{\varepsilon} (\Delta \varepsilon_{ij}^r) dV = \lambda_{UB}^c \int_V \sum_{r=1}^m \hat{\sigma}_{ij}(t_r) \Delta \varepsilon_{ij}^r dV \quad (10)$$

The times t_r correspond to the extremes of the elastic stress histories.

The linear matching method relies upon the generation of a sequence of linear problems where the moduli of the linear problem are found by a matching process. For the von Mises yield condition the appropriate class of strain rates chosen are incompressible and the linear problem is defined by a single shear modulus $\mu(x_i, t)$ which varies both spatially and during the cycle. Corresponding to an initial estimate of the strain rate history $\dot{\varepsilon}_{ij}^i$, a history of a shear modulus $\mu(x_i, t)$ may be defined by the matching condition,

$$3\mu \bar{\varepsilon}^i = \sigma_y \quad (11)$$

where the matching condition occurs at each instant in the cycle. We now define a corresponding linear problem for a new kinematically admissible strain rate history, $\dot{\varepsilon}_{ij}^f$ and a time constant residual stress field $\bar{\rho}_{ij}^f$.

$$\dot{\varepsilon}_{ij}^f = \frac{1}{2\mu} (\bar{\rho}_{ij}^f + \lambda \hat{\sigma}_{ij}'), \quad \dot{\varepsilon}_{kk}^f = 0 \quad (12)$$

where $\bar{\rho}_{ij}^f$, $\hat{\sigma}_{ij}'$ and $\dot{\varepsilon}_{ij}^f$ represent the deviatoric part of residual, elastic stress and strain tensors, respectively. Integrating (12) over the cycle $0 \leq t \leq \Delta t$ leads to:

$$\Delta \varepsilon_{ij}^f = \frac{1}{2\bar{\mu}} (\bar{\rho}_{ij}^f + \lambda_{UB}^i \sigma_{ij}^{'in}) \quad (13)$$

where we assume $\lambda = \lambda_{UB}^i$ and,

$$\frac{1}{\bar{\mu}} = \int_0^{\Delta t} \frac{1}{\mu(t)} dt \quad \text{and} \quad \sigma_{ij}^{'in} = \bar{\mu} \left\{ \int_0^{\Delta t} \frac{1}{\mu(t)} \hat{\sigma}_{ij}' dt \right\} \quad (14)$$

The solution of this linear problem, Ponter and Engelhardt (2001); Ponter *et al.* (2002), has the property of reducing the upper bound (4), i.e.

$$\lambda_{UB}^f \leq \lambda_{UB}^i \quad (15)$$

where λ_{UB}^i and λ_{UB}^f correspond to the upper bounds (4) derived from $\dot{\varepsilon}_{ij}^i$ and $\dot{\varepsilon}_{ij}^f$ respectively. Equality occurs if and only if $\dot{\varepsilon}_{ij}^i \equiv \dot{\varepsilon}_{ij}^f$. The repeated application of this procedure results in a monotonically reducing sequence of upper bounds that converges to the least upper bound within a class of displacement fields and strain rate histories. A primary objective of the

implementation of the technique is to ensure that the classes chosen results in a minimum that is sufficient close to the absolute minimum to be of practical use as a substitute for an analytic solution.

Again, assuming plastic strains only occur at m instances during the load cycle and replacing the continuous strain rate history $\dot{\varepsilon}_{ij}^c$ by m discrete increments of plastic strain ($\Delta \varepsilon_{ij}^r$, $r=1, \dots, m$), equations (11) (12) and (14) become;

$$3\mu^r \bar{\varepsilon}(\Delta \varepsilon_{ij}^r) = \sigma_y \quad (16)$$

$$\Delta \varepsilon_{ij}^{rf} = \frac{1}{2\mu^r} (\bar{\rho}_{ij}^f + \lambda \hat{\sigma}_{ij}'(t_r)) , \quad \Delta \varepsilon_{kk}^r = 0 \quad \text{and}$$

$$\Delta \varepsilon_{ij}^{rf} = \sum_{r=1}^m \Delta \varepsilon_{ij}^{rf} \quad (17)$$

$$\frac{1}{\bar{\mu}} = \sum_{r=1}^m \frac{1}{\mu^r} \quad \text{and} \quad \sigma_{ij}^{rin} = \bar{\mu} \left\{ \sum_{r=1}^m \frac{1}{\mu^r} \hat{\sigma}_{ij}'(t_r) \right\} \quad (18)$$

The algorithm above has been incorporated into a standard finite element scheme for the solution of linear problems, using the commercial FE code ABAQUS. When the sufficient condition for convergence applies, the method converges to the least upper bound associated with the finite element mesh. The converged solution then satisfies the conditions that the stress history $\bar{\rho}_{ij}^f + \lambda \hat{\sigma}_{ij}'(t_r)$ satisfies the yield condition at the Gauss points where the matching condition is applied, the strain rates are associated with the yield surface when this stress reaches yield

and the strain rates accumulate to a compatible strain distribution. Hence, all conditions for both the upper and lower bound shakedown theorem are satisfied except that the stress distribution satisfied equilibrium in an average sense.

Applications

In this section we apply the above discussed technique to solve some examples and if possible, compare the obtained results with those obtained from other methods.

Portal Frame Under Vertical and Horizontal Loads

We consider the uniform rectangular fixed base portal frame in Fig. 1(a), subjected to a central vertical load and a horizontal load at beam level which are varying independently between zero and the positive values ($H(t)$, $V(t)$).

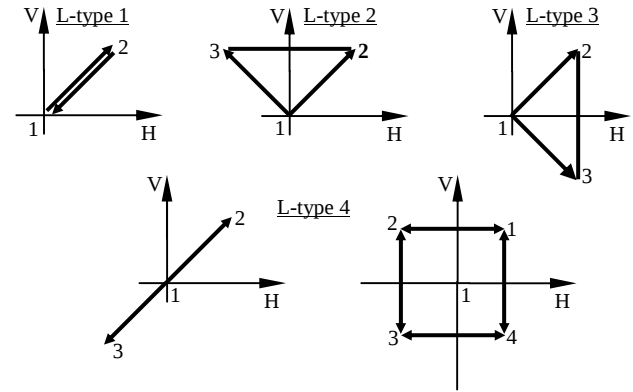


FIG. 2 LOAD HISTORIES

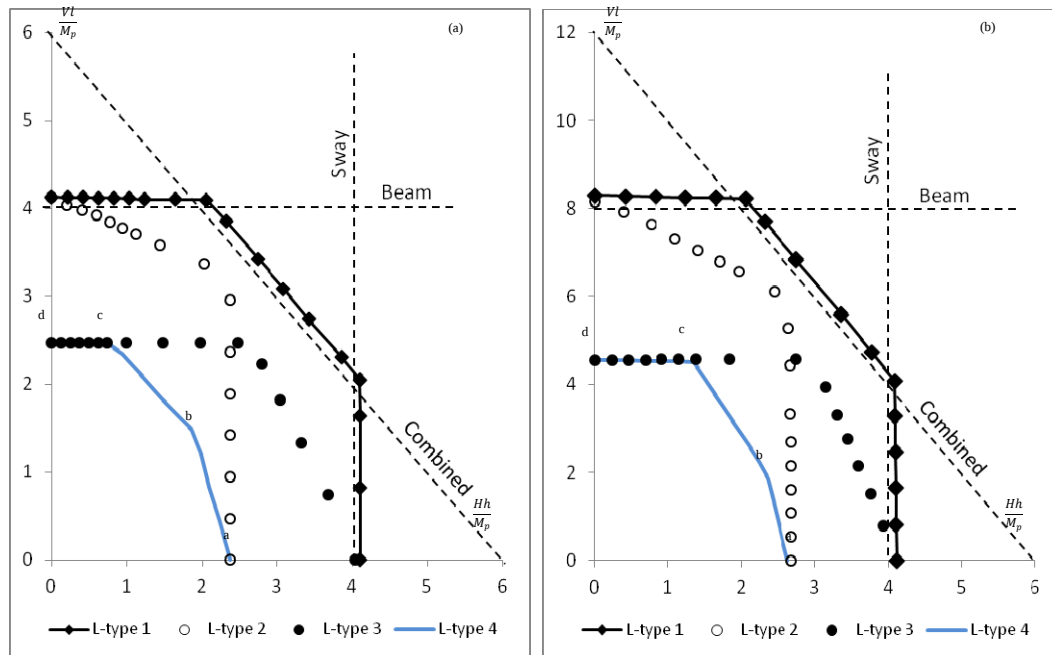


FIG. 3 INTERACTION DIAGRAMS, (a) $h/l=0.5$, (b) $h/l=1.0$

Figure 2 shows a cyclic load history which describes a sequence of straight lines in load space produces a history of elastic stress of a similar form in stress space. For such problems plastic strains are confined to a finite number of instants t_1 to t_4 during the cycle.

The shakedown limits have been evaluated for the four histories of $(H(t), V(t))$ shown in Fig. 2. The interaction diagrams of the shakedown limit evaluated by the proposed method are shown in Fig. 3 together with the limit load when the two independent loads system (H, V) are acting simultaneously on the structure in any ratio. It is worth mentioning that any combinations of H and V represented by a point inside the diagram constitutes a safe state of external loading.

From Figure 3 it can be seen that domains corresponding to Load-type 1 obtained by LMM closely match results obtained by equations 1-3, which identifies the domain for static collapse. This demonstrates the flexibility of the method in producing a range of performance indicators for structures subjected to proportional loadings. It also can be seen that for all load combinations, Load-type 4 defines a permissible region $oabcd$ within which any combination of vertical and horizontal loads will not cause incremental collapse or failure by alternating plasticity.

Portal Frame Under Distributed Pressure

The fixed base portal frame in Fig. 4 is now subjected to a vertical uniform pressure varying between zero and W and a horizontal load at beam level varying

between zero and H . It is assumed that the load variations can take place independently, and we shall explore the problem for all possible combinations of $(W(t), H(t))$ as shown in Fig. 2.

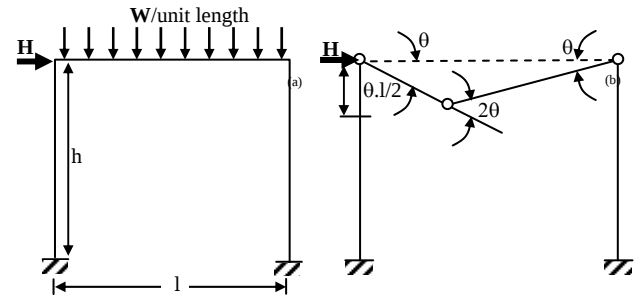


FIG. 4 PORTAL FRAME UNDER HORIZONTAL LOAD AND VERTICAL PRESSURE

Under proportional loading, one can demonstrate that for the beam mechanism which occurs when the vertical load is the dominant loading:

$$Wl / M_p = 16 \quad (19)$$

and for the combined mechanism, the limiting load is given by:

$$Wl / 4M_p + Hh / M_p = 6 \quad (20)$$

The combinations of W and H causing collapse according to the various mechanisms are shown graphically in Fig. 5. The load corresponding to the chosen mechanism is therefore an upper bound on the collapse load of the original frame. The argument applies to incremental collapse as well as to simple plastic collapse under proportional loading.

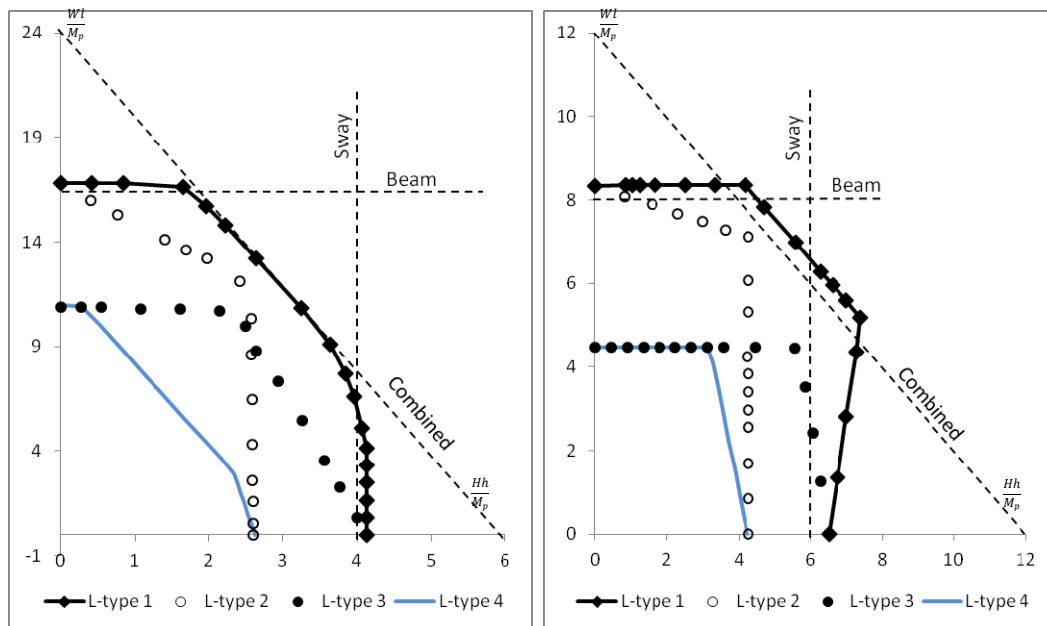


FIG. 5 INTERACTION DIAGRAMS, (a) $h/l=0.5$, (b) $h/l=1.0$

It can be seen from these figures that the predicted domains under load type-1 give very close correlation with domains obtained by plastic analysis for simple plastic collapse under proportional loading.

Conclusions

The proposed Linear Matching Method provides a general-purpose technique to deal with shakedown analysis of simple portal frame structures under repeated and cyclic loads. It overcomes the difficulty of numerical calculation and so the shakedown analysis of structures under various combinations of steady and variable mechanical loading can more readily be performed in practice. The set of examples of both limit loads and shakedown limits given here demonstrate its numerical stability and the ability to approach the analytic solution from above through mesh refinement. Furthermore, its implementation within routines of commercial finite element codes e.g. ABAQUS allows the method to be introduced to more general industrial practices.

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